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ABSOLUTE CONTINUITY IN BANACH SPACE THEORY

BY

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The concept of absolute continuity as described below was first introduced by author in [8] as a direct generalization of the well-known result due to Grothendieck [3] and Bartle, Dunford and Schwartz on the existence of control measures. It turns out that the absolute continuity is a common property for all classical Banach ideals of operators such as of compact operators, of absolutely p-summing operators, or of operators factorizing weakly-compactly through a $C(S)$ space. There are essentially only two classes of absolutely continuous operators : weakly compact operators defined on $C(S)$ spaces and absolutely continuous operators defined on l_1. The first is related to our original point of view while the second seems to be useful in studying the notion of super-reflexivity. An easy consequence of Riesz-Thorin interpolation theorem is that every operator T from a space $L_1(\mu)$ into a space $L_p(\nu)$ with $p \geq 2$ is absolutely continuous but it is not very clear what happens for $1 < p < 2$.
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1. GENERAL THEORY

In the sequel $L(E, F)$ (respectively $\Pi_p(E, F)$, $1 \leqslant p < \infty$) will denote the vector space of all bounded linear operators (respectively of all absolutely p-summing operators in the sense of Pietsch [12]) from the Banach space E into the Banach space F. A seminorm p defined on the Banach space E is called pre-nuclear if $p(\cdot) \leqslant \int | < \cdot, x' > | \, d\mu \, (x')$, where μ denotes a positive Radon measure given on the unit ball $B_{E'}$, of E', endowed with the $\sigma(E', E)$-topology. According to [11], Theorem 2.3.3, p is pre-nuclear if and only if p is equivalent to a seminorm $p_S(\cdot) = \|S(\cdot)\|$ where $S \in \Pi_1(E, F)$.

1.1. DEFINITION. An operator $T \in L(E, F)$ is said to be absolutely continuous with respect to the pre-nuclear seminorm p (i.e., $T \ll p$) if T satisfies the following equivalent conditions :

(AC_1) For every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\|T(\cdot)\| \leqslant \varepsilon \, \|\cdot\| + \delta p(\cdot);$$

(AC_2) For every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\|\cdot\| \leqslant 1, \; p(\cdot) \leqslant \delta \text{ implies } \|T(\cdot)\| \leqslant \varepsilon;$$

(AC_3) Given a sequence $\{x_n\}_n$ of elements of norm 1 of E, then there is either a $C > 0$ such that $\| T(x_n)\| \leqslant Cp(x_n)$, $n \geqslant 1$, or a subsequence $\{x_{n_k}\}_k$ with $T(x_{n_k}) \to 0$.

An operator $T \in L(E, F)$ is called absolutely continuous if $T \ll p_S$ for a suitable $S \in \Pi_1(E, G)$. According to [11], Theorem 2.2.3, S is a product $E \hookrightarrow C(B_{E'}) \hookrightarrow L_1(\mu)$.

The absolutely continuous operators constitute a Banach ideal of operators, say $[AC, \| \|]$, in the sense of Pietsch. Here $\| \|$ denotes the restriction to $AC(E, F)$ of the usual norm in $L(E,F)$. In fact, it is clear that the product of an absolutely continuous operator and a bounded linear mapping is also an absolutely continuous operator. Now let $\{T_n\}_n$ be a Cauchy sequence in $AC(E, F)$ and choose for each $n \in \mathbb{N}$ a probability measure μ_n on $B_{E'}$, such that $T_n \ll p_n$, where $p_n(.) = \int | < ., x' > | \, d\mu_n(x')$. Put $\mu = \Sigma 2^{-n}\mu_n$ and $T = \lim T_n$. Then :

$$\|Tx\| \leqslant \|T_n(x)\| + 2\varepsilon \|x\| \sup_k \|T_k\|$$

$$\leqslant \varepsilon(1 + \sup_k \|T_k\|) \|x\| + \delta_n(\varepsilon) \int | <x, x'>) | d\mu_n(x')|$$

$$\leqslant \varepsilon(1 + \sup_k \|T_k\|) \|x\| + 2^n \delta_n(\varepsilon) \int | < x, x' >| \, d\mu \, (x')$$

for $n = n(\varepsilon)$ sufficiently large, which in turn implies that $T \in AC(E, F)$.

We define for Banach ideals an order relation by $[A, \alpha] \leqslant [B, \beta]$ if and only if $A(E,F) \subset B(E, F)$ and $\alpha(T) \geqslant \beta(T)$ for $T \in A(E,F)$ then $[AC, \| \|]$ is larger then all classical ideals of operators :

1) $[\Pi_p, \pi_p]$, of absolutely p-asumming operators in the sense of Pietsch [12], $1 \leqslant p < \infty$;

2) $[I_p, i_p]$, of p-integral operators in the sense of Persson and Pietsch [10], $1 \leqslant p < \infty$;

3) $[N_p, \nu_p]$, of p-nuclear operators in the sense of Persson and Pietsch [10], $1 \leqslant p < \infty$;

4) $[N_p^Q, \nu_p^Q]$, of quasi-p-nuclear operators in the sense of Persson and Pietsch [10] $1 \leqslant p < \infty$;

5) $[K, \| \|]$, of compact operators;

6) $[C_w, c_w]$, of operators which factorize weakly-compactly through a $C(S)$ space; $T \in C_w(E, F)$ if there are $U \in L(E, C(S))$ and $V \in L(C(S), F)$ such that V is weakly compact and $T = V \circ U$. The norm c_w is given by $c_w(T) = \inf \|U\| \cdot \|V\|$, the infimum over all factorizations $T = V \circ U$.

The main properties of absolutely continuous operators are indicated in Theorem 1.2 below. One of them, namely the stability, is new even for operators acting on $C(S)$ spaces. Recall that a sequence $\{x_n\}_n$ of elements of a Banach space E is said to be stable (with limit x) if there exists an $x \in E$ such that $\left\| \frac{1}{n} \sum_{i \leqslant n} x_{k_i} - x \right\| \to 0$ uniformly in the set of all strictly increasing sequences $\{k_n\}_n$ of natural numbers. This concept comes from the ergodic theory (a measure-preserving point transformation τ is stable if for every $f \in L_2$ the sequence $\{f \circ \tau^n\}_n$ is stable in the Banach space L_2) and was first studied by Brunel and Sucheston [1].

1.2. THEOREM.

(i) *Every absolutely continuous operator* T *is strictly singular i.e., no restriction of* T *to an infinite-dimensional subspace is an isomorphism.*

(ii) (Dunford-Pettis property). *Every absolutely continuous operator maps weak Cauchy sequences into converging sequences.*

(iii) *Every absolutely continuous operator is stable i.e., it maps bounded sequences into sequences with stable subsequences.*

(iv) *Every absolutely continuous operator is weakly compact.*

(v) *The product of two absolutely continuous operators is a compact operator.*

(vi) *An operator* $T \in L(E, F)$ *is absolutely continuous provided that its restriction to any separable subspace of* E *is absolutely continuous.*

Proof. (*i*) Let $T \in AC(E, F)$ with $T \ll S$ and $S \in \Pi_1(E,G)$. If $T|H$ is an isomorphism then there is a $\lambda > 0$ such that $\lambda \|x\| \leqslant \|Tx\| \leqslant (\lambda/2)\|x\| + \delta(\lambda/2)\|Sx\|$ for every $x \in H$, so that $S|H$ is an isomorphism. It remains to recall the result due to Dvoretzki and Rogers (e.g., see [11], Theorem 3.4.1) which asserts that every absolutely summing operator is strictly singular.

(*ii*) Let $T \in AC(E, F)$ with $T \ll p$ where $p(\,.\,) = \int |<., x'>| \, d\mu\,(x')$.

If $\{x_n\}_n$ is a weak Cauchy sequence in E then Lebesgue's theorem on dominated convergence yields that

$$\lim_{m,\, n \to \infty} \int |< x_m - x_n, x' >| \, d\mu\,(x') = 0$$

and thus by (AC_1) it follows that the sequence $\{Tx_n\}_n$ is Cauchy in norm.

(*iii*) If $T \in AC\,(E,F)$ then T satisfies an estimate as follows:

$$\|Tx\| \leqslant \varepsilon\|x\| + \delta\,(\varepsilon)\left(\int |< x,x' >|^2 \, d\mu\,(x')\right)^{1/2}$$

and every bounded sequence of elements $x_n \in E$ can be regarded as a bounded sequence in $C(B_{E'})$ (and thus in $L_2\,(\mu)$) e.g., by using the isometry $x \to \varphi_x(x') = <x,x'>$. A result due to Brunel and Sucheston [1] asserts that every bounded sequence in $L_2(\mu)$ has a stable subsequence and thus it remains to make use of the following estimate (I and J are two finite subsets):

$$\left\| \frac{1}{\text{Card } I} \sum_{i \in I} Tx_i - \frac{1}{\text{Card } J} \sum_{j \in J} Tx_j \right\|$$

$$\leqslant \varepsilon \sup \|x_n\| + \delta(\varepsilon) \left\| \frac{1}{\text{Card } I} \sum_{i \in I} x_i - \frac{1}{\text{Card } J} \sum_{j \in J} x_j \right\|$$

(*iv*) If $T \in L(E,F)$ is not weakly compact then a result due to Lindenstrauss and Pelczynski [7] asserts the existence of two operators, $S_1 \in L(E, l_1)$ and $S_2 \in L(F, l_\infty)$ such that $S_2 \circ T \circ S_1 = \sigma$ where $\sigma : l_1 \to l_\infty$ is given by $\sigma(\{\lambda_n\}_n) = \left\{ \sum_{i=1}^{n} \lambda_i \right\}_n$. Because σ is not stable, it follows (according to (*iii*)) that T cannot be absolutely continuous.

The assertion (*v*) follows now from (*ii*) and (*iv*) .

(*vi*). From hypotheses follows easily the existence of a function $\delta : (0, \infty) \to [0, \infty)$ such that for every finite-dimensional subspace $G \subset E$ there is a probability measure μ_G on $B_{E'}$, with $\|Tx\| \leqslant \varepsilon \|x\| + \delta(\varepsilon) \int | < x, x' > | \, d\,\mu_G(x')$ for every $x \in G$. The set $\{\mu_G\}_G$ such obtained has a weak limit point, say μ, and it is clear that T is absolutely continuous with respect to the pre-nuclear seminorm generated by μ, q.e.d.

Even the adjoint of an absolutely summing operator is not necessarily absolutely continuous, e.g., the adjoint of the canonical inclusion of l_1 into l_2 does not obey (*ii*) in Theorem 1.2 above. The case of the second adjoint is treated in the following

1.3. THEOREM. *Let* $T_i \in L(E, F_i)$, $i = 1 ; 2$. *Then*

$$T_1 \ll p_{T_2'} \text{ if and only if } T_1'' \ll p_{T_2''}.$$

The proof is a consequence of the following :

1.4. LEMMA. *Let* $S \in L(E, F)$, *let* $H \subset E'$ *be a finite-dimensional subspace,* $x'' \in E''$, *and let* $\eta > 0$. *Then there exists an* $x \in E$ *such that*

(*i*) $\|x\| \leqslant (1 + \eta) \|x''\|$,

(*ii*) $\|Sx\| \leqslant (1 + \eta) (\|S''x''\| + \eta)$,

(*iii*) $<x, x'> = < x'', x' >$, $x' \in H$.

The proof below was suggested to us by D. R. Lewis.

Clearly, we can assume that $x'' \neq 0$. Put $\alpha = \|x''\| / \|S''x''\|$ if $S''x'' \neq 0$ and $\alpha = \|x''\|/\eta$ otherwise.

If Graf αS is normed by

$$\|\{ \cdot, \cdot \}\| = \max \{ \| \cdot \|, \| \cdot \| \}$$

then Graf $\alpha S''$ is a closed subspace of (Graf $\alpha S)''$ and we shall apply the principle of local reflexivity (see [6]) for

$$\{x'', \alpha S'' x''\} \in (\text{Graf } \alpha S)''$$

$$\{x', 0\} \in (\text{Graf } \alpha S)', \quad x' \in H$$

and for $\eta > 0$ given above. Then there exists a pair $\{x, \alpha Sx\} \in$ Graf αS such that

$$(iv) \quad \|\{x, \alpha Sx\}\| \leqslant (1+\eta)\, \|\{x'', \alpha S''x''\}\| = (1+\eta)\|x''\|$$

$$(v) \quad <\{x, \alpha Sx\}, \{x', 0\}> = <\{x'', \alpha S''x''\}, \{x', 0\}>, \; x' \in H.$$

If $S''x'' \neq 0$ then $\|x''\| = \alpha\|S''x''\|$ and thus (iv) implies that $\|x\| \leqslant (1+\eta)\,\|x''\|$ and $\|Sx\| \leqslant (1+\eta)\ \|S''x''\|$. If $S''x'' = 0$ then from (iv) it follows that $\|Sx\| \leqslant \eta\,(1+\eta)$ and again $\|x\| \leqslant (1+\eta)\|x''\|$. On the other hand, (v) yields immediately that $<x, x'> = <x'', x'>$ for all $x' \in H$, q.e.d.

Proof of Theorem 1.3. We shall show only that $T_1 \ll p_{T_2}$ implies $T_1'' \ll p_{T_2''}$. For, let us consider an $x'' \in E''$, a $y' \in F_1'$, $\|y'\| \leqslant 1$ and an $\eta > 0$. By (AC_1):

$$\|T_1(.)\| \leqslant \varepsilon\|.\| + \delta(\varepsilon)\,\|T_2(.)\|.$$

Let $x \in E$ as in Lemma 1.4 (applied for $H = \mathbb{C}\cdot T_1'y'$ and $S = T_2$). Then:

$$\begin{aligned}|<T_1''x'', y'>| &= |<x, T_1'y'>| = |<x, T_1'y'>|\\ &= |<T_1x, y'>| \leqslant \|T_1x\|\\ &\leqslant \varepsilon\,\|x\| + \delta(\varepsilon)\|T_2x\|\\ &\leqslant \varepsilon(1+\eta)\,\|x''\| + \delta(\varepsilon).\,(1+\eta)\,(\|T_2''x''\| + \eta),\end{aligned}$$

q.e.d.

2. EXAMPLES OF ABSOLUTELY CONTINUOUS OPERATORS

First notice that $\Pi_p(E,F) \subset AC(E,F)$, $1 \leqslant p < \infty$, for all Banach spaces E, F. Indeed, each $T \in \Pi_p(E,F)$ satisfies an estimate as follows:

$$\begin{aligned}\|Tx\| &\leqslant \pi_p(T)\left(\int |<x, x'>|^p \,d\mu(x')\right)^{1/p}\\ &\leqslant (\varepsilon\|x\|)^{1/q}\left[\pi_p(T)^p\,\varepsilon^{1-p}\int |<x, x'>|\,d\mu\,(x')\right]^{1/p}\\ &\leqslant \frac{\varepsilon}{q}\,\|x\| + \frac{\pi_p(T)^p}{p\,\varepsilon^{p-1}}\int |<x, x'>|\,d\mu(x')\end{aligned}$$

where $1/p + 1/q = 1$.

Our next result shows that $AC(E,F)$ contains also all compact operators $T \in L(E,F)$:

2.1 THEOREM. *An operator $T \in L(E,F)$ is compact if and only if T is absolutely continuous with respect to a nuclear operator.*

Proof. If $T \ll p_S$ for a suitable $S \in N(E, G)$, then for every bounded sequence of elements $x_n \in E$ we have

$$\|Tx_m - Tx_n\| \leqslant \varepsilon \|x_m - x_n\| + \delta(\varepsilon) \|Sx_m - Sx_n)\|$$

for all $m, n \geqslant 1$. Because S is compact we can assume $\{Sx_n\}_n$ to be convergent and thus the above estimate easily yields that T is also compact.

Conversely, let $T \in K(E, F)$ and let $i : F \to l_\infty(B_{F'})$ be the canonical isometry given by $i(x) = \{x'(x)\}_{\|x'\| \leqslant 1}$. By [5], p. 23, $i \circ T$ has a compact extension $\tilde{T} \in L(C, (B_{E'}), l_\infty(\beta_{F'}))$. According to [14], page 232, $|\tilde{T}|$ exists and is compact, so that $|\tilde{T}|$ can be approximated by positive operators $S_n = \sum_{i=1}^{N(n)} x'_{ni} \otimes y_{ni}$ of finite rank. Put

$$S = \Sigma\, 2^{-n}(1 + \nu_1(S_n))^{-1} S_n.$$

Then $S \in N(C(B_{E'}), l_\infty(B_{F'}))$ and $S > 0$. Moreover, for every $x \in C(B_{E'})$, $x > 0$:

$$\|\tilde{T}x\| \leqslant \||\tilde{T}|x\| \leqslant \|(|\tilde{T}| - S_n)\, x\| + \|S_n(x)\|$$

$$\leqslant \||\tilde{T}| - S_n\| . \|x\| + 2^n(1 + \nu_1(S_n)) \|S_x\|$$

which in turn implies that $\tilde{T} \ll p_S$, q.e.d.

The result above can be strengthened as follows :

2.2. THEOREM. *Let E be a Banach space that contains no isomorphic copy of l_1 and let T be a bounded linear mapping from E into the Banach space F. The following assertions are equivalent :*

i) T is compact;

ii) T maps weakly converging sequences into norm-converging sequences;

iii) T is absolutely continuous.

Proof. The implication $i) \Rightarrow iii)$ follows from Theorem 2.1 above, while $iii) \Rightarrow ii)$ is an easy consequence of Theorem 1.2.

$ii) \Rightarrow i)$. According to [13], each bounded sequence of elements $x_n \in E$ has a weak Cauchy subsequence, say $\{y_n\}_n$. If $\{p_n\}_n$ and $\{q_n\}_n$ are two strictly increasing sequences, $\{y_{p_n} - y_{q_n}\}_n$ is weakly converging to 0 and thus $\|Ty_{p_n} - Ty_{q_n}\| \to 0$, q.e.d.

The result above suggests that $\Pi_1(E,F) = N^Q(E,F)$ for every Banach space which contains no isomorph of l_1. That is true at least where E' is complemented in a Banach lattice (i.e., E has local unconditional structure in the terminology given by Gordon and Lewis [2]). For the proof, make use of Theorem 7 in [9] which asserts that under the above hypotheses E' has the Radon-Nikodym property.

Our next result shows that $C_w(E,F) \subset AC(E, F)$.

2.3 THEOREM*. *Let E be a Banach space whose dual is complemented in an $L_1(\mu)$ space and let T be a bounded operator from E into the Banach space F. The following assertions are equivalent :*

i) T *is weakly compact* ;

ii) T *is absolutely continuous* ;

iii) T *is stable.*

Proof. By Theorem 1.2 we have only to prove that *i)* $\Rightarrow$ *ii)* and we start with the case where E is a space $C(S)$. Then $\mathscr{A} = \{ |y' \circ T| \, ; \, y' \in B_{F'}\}$ is a weakly relatively compact subset of $C(S)'$. (See [14], page 119). Let us denote by $M(S)$ the Banach lattice of all Borel measurable bounded functions $f : S \to \mathbb{R}$.

Claim. For every $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ and a finite subset $\mathscr{A}_\varepsilon \subset \mathscr{A}$ such that if $f \in M(S)$, $0 \leqslant f \leqslant 1$ and $\sup_{\mu \in \mathscr{A}_\varepsilon} \int f \, d\mu < \delta$ then $\sup_{\mu \in \mathscr{A}} \int f \, d\mu < \varepsilon$.

In fact, if the contrary is true, then there are an $\varepsilon_0 > 0$, a sequence $0 \leqslant f_n \leqslant 1$ in $M(S)$ and a sequence $\mu_n \in \mathscr{A}$ such that

$$\text{a)} \qquad \sup_{1 \leqslant k \leqslant n} \int f_n \, d\mu_k \leqslant 2^{-n-1}$$

$$\text{b)} \qquad \int f_n \, d\mu_{n+1} \geqslant \varepsilon_0$$

for all $n \geqslant 1$. Put $g_n = \sup \{f_k \, ; \, k \geqslant n\}$ and $g = \inf \{g_n \, ; \; n \geqslant 1\}$. Then

$$\text{a}') \qquad \sup_{1 \leqslant k \leqslant n} \int g_n \, d\mu_k \leqslant 2^{-n}$$

for all $n \geqslant 1$, which implies that

$$\int g \, d\mu_k = \lim_{n \to \infty} \int g_n \, d\mu_k = 0$$

uniformly for $k \in \mathbb{N}$ (use Theorem 2 in [3]), which contradicts b).

Then T is absolutely continuous with respect to the pre-nuclear seminorm associated to

$$\mu = \sum_n \, (2^n \text{ Card } \mathscr{A}_{1/n})^{-1} \sum_{\lambda \in \mathscr{A}_{1/n}} \lambda.$$

In the general case T'' has a weakly compact extension to a space $L_\infty(\mu)$ and as remarked above there exists an $S \in \Pi_1 (L_\infty(\mu), G)$ such that $T'' \ll S$, which implies that $T \ll S|E$, q.e.d.

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* See also the author's paper: *Weakly compact operators on C*-algebras*, Rev. Roum. Math. Pures et Appl., to appear.

We turn now to absolutely continuous operators defined on a space $L_1(\mu)$.

A classical result due to Grothendieck [5] asserts that $L(L_1(\mu), L_2(\nu)) = AC(L_1(\mu), L_2(\nu))$ for all positive Radon measures μ and ν. By combining Riesz-Thorin interpolation theorem with Theorem 1.2 above we obtain that $L(L_1(\mu), L_p(\nu)) = AC(L_1(\mu), L_p(\nu))$ for all $p \geqslant 2$.

Each $T \in L(l_1, E)$ is associated to a bounded sequence of elements $x_n \in E$ as follows : $Te_n = x_n$, $n \geqslant 1$, where $\{e_n\}_n$ constitutes the unit vector basis in l_1. By Theorem 1 in [5], $S \in \Pi_1(l_1, F)$, if and only if S can be factorized through l_2, and thus $T \in AC(l_1, E)$ if and only if there exists a bounded sequence of elements $u_n \in l_2$ such that :

$$\|\Sigma \lambda_n x_n\| \leqslant \varepsilon \Sigma |\lambda_n| + \delta(\varepsilon) \|\Sigma \lambda_n u_n\|_{l_2}$$

whenever $\{\lambda_n\}_n \in l_1$. Consequently, if $\{r_n\}_n$ denotes the Rademacher system on [0,1] then

$$\left\| \sum_{n=1}^{N} r_{k_n}(t)\, x_n \right\| \leqslant \varepsilon N + \delta(\varepsilon) \left\| \sum_{n=1}^{N} r_{k_n}(t)\, h_n \right\|_{l_2}$$

for all $t \in [0, 1]$, $1 \leqslant k_1 < \ldots < k_N$, $N \geqslant 1$. Because the Rademacher functions constitute an orthonormal system in $L_2[0,1]$ it follows that

$$\inf_{1 \leqslant k \leqslant N} \left\| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{N} x_i \right\| \leqslant \varepsilon N + \delta(\varepsilon) N^{1/2}$$

for all $N \geqslant 1$, and thus the following proposition is true.

2.4 Proposition. *If* $L(l_1, E) = AC(l_1, E)$ *then* E *is super-reflexive.*

We do not know whether every absolutely continuous operator can be factorized through a super-reflexive space. Still open is the following.

2.5 *Problem.* Can every operator $T \in AC(E, l_\infty(\Gamma))$ be extended to a weakly compact operator $\tilde{T} \in L(C(B_{E'}), l_\infty(\Gamma))$?

3. FURTHER EXTENSIONS

The concept of absolute continuity as introduced above can be extended in the following manner. Let $\mathscr{I}$ be a Banach ideal of operators and let $\mathscr{D}$ be a cone of functions $\delta : (0, \infty) \to [0, \infty)$. We shall denote by $AC_{\mathscr{I},\mathscr{D}}$ the Banach ideal of all operators T satisfying estimates as follows

$$\|Tx\| \leqslant \varepsilon \|x\| + \delta(\varepsilon) \|Sx\|$$

with $S \in \mathscr{I}$ and $\delta \in \mathscr{D}$.

The proof of Theorem 1.3 yields easily that $T \in AC_{\mathscr{I},\mathscr{D}}$ implies $T'' \in AC_{\mathscr{I},\mathscr{D}}$ provided that $S \in \mathscr{I}$ implies $S'' \in \mathscr{I}$.

If $\mathcal{J}$ is formed by strictly singular operators then each $T \in AC_{\mathcal{J},\mathcal{D}}$ is strictly singular and maps bounded sequences into sequences with weak Cauchy subsequences (use the main result in [13]). If $\mathcal{J}$ is formed by stable operators so is $AC_{\mathcal{J},\mathcal{D}}$.

The conclusion of Theorem 2.3 above can be improved as follows:

3.1 THEOREM. *Let E be a Banach lattice whose dual is weakly sequentially complete and such that E is contained in the band generated by a suitable $u'' \in E''$. If T is a weakly compact operator from E into a Banach space F then:*

(AC) *there exists a $u' \in E'$, $u' > 0$, such that*

$$x \in E, \quad |x| \leqslant x'', \quad u'(|x|) < \delta(\varepsilon, x'') \Rightarrow \|Tx\| < \varepsilon$$

for every $\varepsilon > 0$ and every $x'' \in E''$, $x'' > 0$; and,

(st) *if $\{x_n\}_n$ is a sequence of elements of E which is order-bounded in E'' then $\{Tx_n\}_n$ has a stable subsequence.*

Proof. (AC). Let $(E'')_{x''}$ denote the ideal generated by a positive $x'' \in E''$, endowed with the norm

$$\|y''\|_{x''} = \inf\{\lambda;\ |y''| \leqslant \lambda\, x''\}.$$

Then $(E'')_{x''}$ is order isometric to a Banach lattice $C(K_{x''})$ and Remark 6 in [9] yields the existence of a positive $u' \in E'$ such that

$$0 < y'' \in E'', \quad y''(u') = 0 \quad \text{implies that} \quad T''(y'') = 0,$$

which in turn implies that the weakly compact operator $T''|C(K_{x''})$ is absolutely continuous with respect to the seminorm $p_{u'}(\,.\,) = u'(|\,.\,|)$.

(st). By (AC) we have

$$\|T''\,x\| \leqslant \varepsilon\,\|x\|_{x''} + \delta(\varepsilon, x'')\left(\int_{K_{x''}} |x(k)|^2\, du'(k)\right)^{\frac{1}{2}}$$

for every $\varepsilon > 0$ and every $x \in (E'')_{x''}$ and thus the proof proceeds exactly as in Theorem 1.2 (*iii*) above.

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